Rapid numerical difference recurrent formula of nonlinear Schrödinger equation and its application

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Abstract

In this paper, a rapid numerical difference recurrent formula, in which it has been taken that the chromatic dispersion and the nonlinearity act together along each fiber segment, is established in the time domain by applying a Maclaurin expansion to the differential form of the nonlinear Schrödinger equation (NLSE) in the frequency domain. The calculated results by using the established formula are contrasted with the known analytical results and the results of the split-step Fourier method (SSFM) and indicated that the rapid numerical difference recurrent formula is very accurate and more reasonable because it abandons an assumption that the dispersive and nonlinear effects can be assumed to act independently as the optical field propagates over each fiber segment. It has been concluded that the established formula in this paper is a scientific, reasonable and effective numerical method for the study of light pulse propagation in a nonlinear optical medium.

Keywords: Rapid numerical difference recurrent formula; Maclaurin expansion; Nonlinear Schrödinger equation (NLSE); Optical pulse propagation

1. Introduction

The nonlinear Schrödinger equation (NLSE) is a very important fundamental equation to study optical pulse propagation in a nonlinear dispersion medium. Therefore, exploring the NLSE equation has been the subject of considerable investigations so far [1–6]. But, the equation does not generally lend itself to analytical solutions except for some specific cases in which the inverse scattering method (ISM) [7] can be employed, so finding the solutions of the equation still need to depend on various sorts of numerical methods when the input pulse envelope is arbitrary shape (in most cases the Gaussian pulse is generally adopted). These methods can be classified into two broad categories known as the finite-difference methods and the pseudospectral methods. Nowadays, the method that has been used extensively is the split-step Fourier method (SSFM) that has a faster calculation speed than the other finite-difference methods because it adopted the finite-Fourier transform algorithm (FFT) [8]. Moreover, Chen et al. have initially analyzed the split-step wavelet method (SSWM) based on the wavelet transform algorithm recently [9]. In another paper, Pu and Li introduced a new approximate method named double stage analytical method to solve the equation in the project application problem [10]. Nielson and Tian et al. studied the equation using the differential matrix method (DMM) [11–13]. But, these methods mentioned above obtains an approximation solution by assuming that in propagating the optical field over a small distance \( h \), the dispersive and nonlinear effects can be assumed to act independently. But, this assumption does not accord with the actual condition that the chromatic dispersion and the nonlinearity act together along the length of the fiber.

In this paper, the difference form of the nonlinear Schrödinger equation (NLSE) in the frequency domain

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was analyzed by the Maclaurin expansion, and on the basis of it, a rapid numerical difference recurrent formula of NLSE was established successfully, in which it has been thought that the chromatic dispersion and the nonlinearity act together along the same fiber segment. The calculated results as compared with the diversity of methods mentioned above show that it has enough high precision and high calculating speed. Moreover, the physical prospect is more scientific and reasonable because it abandons an assumption that the dispersive and nonlinear effects can be assumed to act independently in propagating the optical field over each fiber segment.

2. Theoretical analysis

2.1. Derivation of the rapid numerical difference recurrent formula

If the width of an optical pulse is not less than 5 ps, the optical pulse propagation in optical medium including loss, chromatic dispersion and nonlinear effect will follow the nonlinear Schrödinger equation (NLSE) as below:

$$\frac{\partial A}{\partial z} + \frac{x}{2} \cdot A + \frac{\beta_2}{2} \cdot \frac{\partial^2 A}{\partial T^2} - \frac{\beta_3}{6} \cdot \frac{\partial^3 A}{\partial T^3} = i\gamma |A|^2 A$$

(1)

where \(A \) is \(A(z, T)\) that represents the slowly varying envelope of the optical pulse. \(z\) is the spatial coordinate that means the distance of transmission. \(T\) means the temporal coordinate in the so-called retarded frame that moves at the speed of the group velocity, represents \(T = t - z/v_g\), and \(v_g\) means the group velocity. The rest parameters \(x, \beta_2, \beta_3, \gamma\) represent the attenuation factor, the second order dispersive parameter, the third order dispersive parameter, nonlinear Kerr effect coefficient, respectively. In order to get the numerical difference recurrent formula in time domain of Eq. (1), firstly it should have the Fourier transform \(A(z, T) \rightarrow \tilde{A}(z, \omega)\). As the style of the Fourier transform of the first order partial differential derivation on \(T\) should be represented as \(\frac{\partial A}{\partial T} \rightarrow -i\omega \tilde{A}(z, \omega)\) [14], one can obtain the expression as below

$$\frac{\partial \tilde{A}}{\partial z} + \frac{x}{2} \cdot \tilde{A} - \frac{\beta_2}{2} \cdot \frac{\partial^2 \tilde{A}}{\partial \omega^2} - \frac{\beta_3}{6} \cdot \frac{\partial^3 \tilde{A}}{\partial \omega^3} = i\gamma |\tilde{A}|^2 \tilde{A}$$

(2)

If the spatial step interval \(\Delta z\) is minute, the calculation of the first order differential derivation can be replaced by its discrete difference form [15], and the following expression can be obtained:

$$\tilde{A}(z + \Delta z, \omega) \approx \left\{1 - \left[\frac{x}{2} - \frac{\beta_2}{2} \cdot \frac{\partial^2 \tilde{A}}{\partial \omega^2} - \frac{\beta_3}{6} \cdot \frac{\partial^3 \tilde{A}}{\partial \omega^3}\right] \cdot \Delta z\right\} \cdot \tilde{A}(z, \omega) + i\gamma \mathcal{F} \left[|\tilde{A}|^2\tilde{A}\right] \cdot \Delta z$$

(3)

Applying the identity of Maclaurin expansion near zero neighborhood [16] for Eq. (3), as the higher-order terms from the second-order upward are negligible at the given step interval \(\Delta z\), the following approximate expression can be obtained:

$$e^{-\left[\frac{x}{2} \cdot \frac{\partial^2 \tilde{A}}{\partial z^2} - \frac{\beta_2}{2} \cdot \frac{\partial^2 \tilde{A}}{\partial \omega^2} - \frac{\beta_3}{6} \cdot \frac{\partial^3 \tilde{A}}{\partial \omega^3}\right] \cdot \Delta z}$$

(4)

By substituting Eq. (4) into Eq. (3), the expression will be acquired as following:

$$\tilde{A}(z + \Delta z, \omega) = e^{-\left[\frac{x}{2} \cdot \frac{\partial^2 \tilde{A}}{\partial z^2} - \frac{\beta_2}{2} \cdot \frac{\partial^2 \tilde{A}}{\partial \omega^2} - \frac{\beta_3}{6} \cdot \frac{\partial^3 \tilde{A}}{\partial \omega^3}\right] \cdot \Delta z} \cdot \tilde{A}(z, \omega) + i\gamma \mathcal{F} \left[|\tilde{A}|^2\tilde{A}\right] \cdot \Delta z$$

(5)

Making the inverse Fourier transform (IFFT) operation to Eq. (5), and the expression of time domain is showed by

$$A(z + \Delta z, T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\left[\frac{x}{2} \cdot \frac{\partial^2 \tilde{A}}{\partial z^2} - \frac{\beta_2}{2} \cdot \frac{\partial^2 \tilde{A}}{\partial \omega^2} - \frac{\beta_3}{6} \cdot \frac{\partial^3 \tilde{A}}{\partial \omega^3}\right] \cdot \Delta z} \cdot \tilde{A}(z, \omega)$$

$$\cdot e^{-i\omega T} \cdot d\omega + i\gamma |\tilde{A}|^2 \tilde{A} \cdot \Delta z$$

(6)

If the spatial coordinate is taken as \(z = k\Delta z, k = 0, 1, 2, 3, \ldots\), then the equation form above can be written by

$$A_{k+1} = h_F \ast A_k + i\gamma |A_k|^2 A_k \cdot \Delta z$$

(7)

where \(h_F\) is given by

$$h_F = h_T(\Delta z, T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\left[\frac{x}{2} \cdot \frac{\partial^2 \tilde{A}}{\partial z^2} - \frac{\beta_2}{2} \cdot \frac{\partial^2 \tilde{A}}{\partial \omega^2} - \frac{\beta_3}{6} \cdot \frac{\partial^3 \tilde{A}}{\partial \omega^3}\right] \cdot \Delta z} \cdot e^{-i\omega T} \cdot d\omega$$

(8)

This expression above can be comprehended as the impulse response function in the absence of the nonlinear effect along the distance \(\Delta z\). Especially, if the third order dispersion coefficient is zero, the analytical solution of Eq. (8) in time domain can be acquired by the integration theory analysis [17] and can be showed as following:

$$h_T(\Delta z, T) = \left[\left[1 + \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\left[\frac{x}{2} \cdot \frac{\partial^2 \tilde{A}}{\partial z^2} - \frac{\beta_2}{2} \cdot \frac{\partial^2 \tilde{A}}{\partial \omega^2} - \frac{\beta_3}{6} \cdot \frac{\partial^3 \tilde{A}}{\partial \omega^3}\right] \cdot \Delta z} \cdot e^{-i\omega T} \cdot d\omega\right] \right] \cdot e^{-i\Delta z}$$

$$-\infty < T < +\infty$$

(9)

Eq. (7) is the fundamental relation of the numerical difference recurrent formula, and by using the relation the correlative operation on the research of the optical pulse propagation can be made. Because it is directly going in the time domain, it is very convenient to use it. Moreover, another form style of Eq. (7) can be obtained by using the following approximation according to the Maclaurin formula when the step interval \(\Delta z\) is small enough

$$A_k e^{i\gamma |A_k|^2 \Delta z} \approx A_k + i\gamma |A_k|^2 A_k \cdot \Delta z$$

So Eq. (7) can also be written as following:

$$A_{k+1} = h_T \ast A_k + i\gamma |A_k|^2 A_k \cdot \Delta z$$

(10)

In the difference recurrent relation shown in Eqs. (7) and (10), the envelope amplitude of the original input optical pulse is defined as

$$A_k|_{k=0} = A_0 = A(0, T)$$

(11)

This expression above is the arbitrary and suitable function of temporal variable \(T\).
2.2. Relation between rapid numerical difference recurrent formula and the known analytical results

Because it has been thought that the chromatic dispersion and the nonlinearity act together along each fiber segment in Eqs. (7) and (10), it is not necessary to separate the process of chromatic dispersion from the nonlinear process to dispose respectively such as the split-step Fourier method (SSFM) when the envelope of optical pulse in the current fiber segment is calculated. So the method proposed above compared with the SSFM is more conformable to the actual situation of the optical pulse propagation in optical medium and its physical prospect is more scientific and reasonable.

It is easy to deduce the analytical expression of the amplitude of the pulse envelope from the analysis mentioned above in the absence of the nonlinear effect. For example, if the dispersion acts alone, i.e., the nonlinear Kerr coefficient is zero ($\gamma = 0$), one can obtain the expression as following:

$$A_{k+1} = h_\beta * A_k = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \left( \frac{j \xi^2 + j \eta^2}{\xi^2} \right)} \Delta_\xi \cdot e^{-i\omega T} \cdot d\omega * A_k$$

(12)

By using the condition of the initial input optical pulse in Eq. (12), after several steps of the convolution operation, we obtain an expression as following:

$$A_{k+1} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \left( \frac{j \xi^2 + j \eta^2}{\xi^2} \right)} \Delta_\xi \cdot e^{-i\omega T} \cdot d\omega * \ldots * \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \left( \frac{j \xi^2 + j \eta^2}{\xi^2} \right)} \Delta_\xi \cdot e^{-i\omega T} \cdot d\omega * A_0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \left( \frac{j \xi^2 + j \eta^2}{\xi^2} \right)} (k+1)\Delta_\xi \cdot e^{-i\omega T} \cdot d\omega * A_0$$

This means the following expression comes into existence permanently:

$$A(z, T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \left( \frac{j \xi^2 + j \eta^2}{\xi^2} \right)} z \cdot \tilde{A}_0 \cdot e^{-i\omega T} \cdot d\omega$$

(13)

Here $\tilde{A}_0$ is the Fourier transform of input optical pulse $A(0, T)$. Obviously, Eq. (13) is the known analytical result of the NLSE equation in the absence of nonlinear effect [18]. We can also obtain the corresponding expression if the nonlinearity acts alone. In order to accomplish the convolution operation in Eqs. (7) and (10), the finite-Fourier transform (FFT) algorithm can be adopted, certainly the wavelet transform algorithm can also be used for its fast speed of transposition and decomposition according to Ref. [9].

For the reason of convenience during the numerical calculation, the normalization processing is made to $A(z, T)$, $z$ and $T$ as

$$A(z, T) = \sqrt{P_0}U(z_0, \tau), \quad \tau = \frac{T}{T_0} \quad z_0 = \frac{z}{L_D}$$

(14)

$$L_D = \frac{T_0^2}{|\beta_2|}, \quad L_N = \frac{1}{\gamma P_0}, \quad N^2 = \frac{L_D}{L_N}$$

(15)

where $L_D$ is the dispersion length and $L_N$ is the nonlinear length. By using Eqs. (14) and (15), $U$ is found to satisfy

$$\frac{\partial U}{\partial z_0} + \frac{L_D}{2} U + \frac{i \text{sgn}(\beta_2)}{2} \frac{\partial^2 U}{\partial \tau^2} - \frac{\text{sgn}(\beta_3)}{6T_0} |\beta_3| \frac{\partial^3 U}{\partial \tau^3} = iN^2 |U|^2 U$$

(16)

where $\text{sgn}(\cdot)$ represents signum function, and the corresponding Gaussian input optical pulse that is included in Eq. (16) is given by

$$U(0, \tau) = e^{-\frac{|\tau| \Delta \tau^2}{2\gamma}}$$

(17)

where $\gamma$ is a chirp parameter. $U(z_0, \tau)$ can be calculated one by one according to the rapid numerical difference recurrent formula represented by Eqs. (7) and (10), thus the corresponding result of the expression (1) is of the form

$$A(z, T) = \sqrt{P_0}U(L_Dz_0, T_0\tau).$$

(18)

3. Calculated results and analysis

In order to validate the availability of Eqs. (7) and (10), the same parameters with Ref. [9] will be selected in the following calculation.

3.1. Case of the dispersion acting alone

The propagation of the unchirped Gaussian optical pulse in the conventional lossless single mode fiber (SMF) are studied by setting $\gamma = 0$ and $C = 0$ and the parameters are substituted correspondingly as: the pulse width is $T_0 = 20$ ps, the pulse peak power is $P_0 = 10$ mW, the fiber loss is $\xi = 0$ dB/km, the second order dispersion coefficient is $\beta_2 = -4$ ps$^2$/km, the third order dispersion coefficient is $\beta_3 = 0$, the transmission distances are $z = 100$ km, 200 km, 300 km, respectively, and the step interval during the process of calculation is $\Delta z_0 = 0.001$, which is equal to $\Delta z = 0.1$ km. The calculated results by using Eq. (7) or Eq. (10) are shown in Fig. 1.

It can be seen from the Fig. 1 that the numerical results calculated by the proposed method in this paper tally with the classical analytical results very well. The maximal error calculated is only 0.03%, the time-consuming in calculation is only 41 s, and this means that the evolution of the optical pulse in the optical medium can be studied by the rapid
3.2. Case of the dispersion and nonlinearity acting together

Considering the case of the dispersion and nonlinearity acting together, the basic parameters are provided as: the pulse width $T_{0} = 20$ ps, the chirp parameter $C = 1$, the pulse peak power $P_{0} = 10$ mW, the fiber loss $a = 0$ dB/km, the second order dispersion coefficient $b_{2} = -4 / C_{0}$ ps$^2$/km, the third order dispersion coefficient still $b_{3} = 0$, the nonlinear Kerr effect coefficient $c = 1$ W$^{-1}$ km$^{-1}$, the transmission distances $z = 200$ km, 300 km, respectively, and the step interval during the calculation $D_{z} = 0.001$, which is equal to $\Delta z = 0.1$ km. The calculated numerical results are shown in Fig. 2.

In Fig. 2, the calculated results by using the rapid numerical difference recurrent method proposed in this paper have been compared with the results of the split-step Fourier method (SSFM) and it can be obtained that the aberration error is not exceeding 0.03%. This means the method established in this paper is very effective for the study of the evolution of the optical pulse in the nonlinear dispersion medium.

3.3. The relation of broadening factor of Gaussian pulse vs. transmission distance

We have also analyzed the broadening factor of a chirped Gaussian pulse along the transmission distance $z$ by using the rapid numerical difference recurrent formula for four different conditions listed as following:

(a) $a = 0$ dB/km, $b_{2} = -4$ ps$^2$/km, $\gamma = 0$ W$^{-1}$ km$^{-1}$, $C = 0$,
(b) $a = 0$ dB/km, $b_{2} = -4$ ps$^2$/km, $\gamma = 0$ W$^{-1}$ km$^{-1}$, $C = 1$,
(c) $a = 0$ dB/km, $b_{2} = -4$ ps$^2$/km, $\gamma = 1$ W$^{-1}$ km$^{-1}$, $C = 1$,
(d) $a = 0.2$ dB/km, $b_{2} = -4$ ps$^2$/km, $\gamma = 1$ W$^{-1}$ km$^{-1}$, $C = 1$.

Especially, the analytical result of the broadening factor with the same parameters of condition (b) listed above is given as following by Ref. [18]:

\[
\text{Broadening factor} = \frac{T_{1}}{T_{0}} = \left[ \left( 1 + \frac{C b_{2} z}{T_{0}^{2}} \right) + \left( \frac{b_{2} z}{T_{0}} \right)^{2} \right]^{1/2}
\]  

In four different conditions listed above, the pulse width $T_{0}$ is set as 20 ps, and the pulse peak power $P_{0}$ is always equal to 10 mW. The calculated results have been shown in Fig. 3.
From the curves shown in Fig. 3, it has been shown that the Gaussian pulse without chirp and nonlinear effect will broaden monotonically with the transmission distance $z$. When the pulse has chirp and $\beta_2 C < 0$, which means that $\beta_2$ and $C$ have different signs, the pulse width goes through an initial narrowing process. At first the pulse width becomes minimum at a special distance, then it begins to broaden monotonically along the transmission distance. For the condition of case (b), it can be found that the curve of the numerical result (marked with “▼”) is coincident well with the analytical result calculated by using Eq. (19). It shows that the rapid numerical difference recurrent formula established in this paper is a quite accuracy method for this kind of calculation. Even when the optics fiber has nonlinear effect, the pulse width will also have a narrow phenomenon. If a chirped Gaussian pulse is transmitted in a lossless dispersive and nonlinear fiber, the broadening factor of the pulse width will change slowly along the distance than the other cases of pulse propagation. And the impact of the fiber loss on the nonlinear effect is more serious than that of other factors during the pulse propagation in the optical medium, because the nonlinear effect section is always proportional to the instantaneous power of the pulse.

4. Conclusion

The difference form of the nonlinear Schrödinger equation (NLSE) is analyzed by employing a Maclaurin expansion in the frequency domain and a rapid numerical difference recurrent formula, in which it has been assumed that the chromatic dispersion and the nonlinearity act together along each fiber segment, is proposed in the time domain. The comparison between the calculated results by applying the established formula in this paper and the calculated results by using the known analytical results and the split-step Fourier method (SSFM) shows that the rapid numerical difference recurrent formula is very accurate and more reasonable because it abandons an assumption that the dispersive and nonlinear effects can be assumed to act independently as the optical field propagates over each fiber segment. Therefore, it has been concluded that the established formula in this paper is a scientific, reasonable and effective numerical method for the study of light pulse propagation in an optical medium. The pulse shape, power, phase, chirp and spectrum can also be calculated after the complex amplitude of the optical pulse is obtained by applying the established formula.

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